

On the metric hypercomplex group alternative-elastic algebras for $n \bmod 8 = 0$.

K.V. Andreev

In this article the fine-tuning of the algorithm 9.1 [2], [3] is considered. In this connection, answers to the following questions are given.

1. How to construct the metric hypercomplex Cayley-Dickson algebra by means of the algorithm 9.1 for $n = 2^k$?
2. How to construct the metric hypercomplex orthogonal group alternative-elastic algebra by means of the algorithm 9.1 for $n \bmod 8 = 0$?
3. How to decompose the metric hypercomplex orthogonal homogenous group alternative-elastic algebra on an algebraic basis?
4. That is the generator of the metric hypercomplex orthogonal homogenous group alternative-elastic algebra and as it to construct?
5. How technically to realize the algorithm 9.1 and to construct the canonical sedenion algebra for $n=16$?

Let product of elements of the hypercomplex group alternative-elastic algebra \mathbb{A} [1], [5], [2], [3] over the field \mathbb{R} and the vector space \mathbb{R}^n ($n \bmod 8=0$) defined as

Axiom 1. *For any elements a and b , their product c is uniquely defined:*

$$c = ab.$$

Axiom 2. *There exists the unique identity element e . For any element a :*

$$ae = ea = a.$$

Axiom 3. *For any element $a \neq 0$, the inverse element a^{-1} is uniquely determined:*

$$aa^{-1} = a^{-1}a = e.$$

Axiom 4. *(The weakly alternative identity.) For any elements a and b :*

$$(aa)b - a(ab) = b(aa) - (ba)a.$$

Axiom 5. *(The flexible identity.) For any elements a and b :*

$$a(ba) = (ab)a.$$

Axiom 6. *(The distributive identity.) For any elements a, b and c :*

$$a(b + c) = ab + ac, (b + c)a = ba + ca.$$

Build one of such the algebras. Suppose that the vector space \mathbb{R}^n is equipped with the metric $(i, j, \dots = \overline{1, n})$

$$\langle a, b \rangle e := \frac{1}{2}(a\bar{b} + b\bar{a}), \quad a = a_0e + \sum_{r=1}^{n-1} a_r e_r, \quad \bar{a} = a_0e - \sum_{r=1}^{n-1} a_r e_r, \quad (1)$$

Let's consider that the metric g_{ij} is Euclidean metric: $\langle a, b \rangle = \delta_{ij}a^i b^j$ in the special orthogonal basis.

$$\begin{aligned} a^{-1} &= \frac{\bar{a}}{\langle a, a \rangle} \quad \forall a \neq 0, \\ a + \bar{a} &= 2a_0e = 2 \langle a, e \rangle e, \\ a(a + \bar{a}) &= 2 \langle a, e \rangle a, \\ a^2 &= - \langle a, a \rangle e + 2 \langle a, e \rangle a, \\ (a + b)^2 &= a^2 + b^2 + ab + ba = -(\langle a, a \rangle + \langle b, b \rangle + 2 \langle a, b \rangle)e + \\ &\quad + 2(\langle a, e \rangle a + \langle a, e \rangle b + \langle b, e \rangle a + \langle b, e \rangle b), \\ \frac{1}{2}(ab + ba) &= - \langle a, b \rangle e + \langle a, e \rangle b + \langle b, e \rangle a. \end{aligned} \quad (2)$$

Definition 1. Let the identities

$$\frac{1}{2}(ab + ba) = - \langle a, b \rangle e + \langle a, e \rangle b + \langle b, e \rangle a, \quad \eta_{(ij)}^k = 2\left(\frac{1}{\sqrt{2}}\eta_{(i)}\delta_j\right)^k - g_{ij}\left(\frac{1}{\sqrt{2}}\eta^k\right), \quad (3a)$$

$$\frac{1}{2} \langle ab - ba, a \rangle = 0, \quad \eta_{(i|j|k)} - \eta_{j(ik)} = 0, \quad (3b)$$

$$\frac{1}{2} \langle ab - ba, e \rangle = 0, \quad \eta_{[ij]}^k \left(\frac{1}{\sqrt{2}}\eta_k\right) = 0. \quad (3b')$$

are executed by definition for all a and b , where η_{ij}^k are the structural constants of the unital algebra with the algebra identity $e = \frac{1}{\sqrt{2}}\eta^k$ and the distributive identity. Such the algebra we name **the orthogonal hypercomplex algebra \mathbb{A}** .

Theorem 1. The orthogonal hypercomplex algebra will be the metric hypercomplex group alternative-elastic algebra.

Proof. The equations (3) are a consequence of the Clifford equation for $n \bmod 8 = 0$ [2], [3] ($A, B, \dots = \overline{1, N}$, $N = 2^{\frac{n}{2}-1}$)

$$\eta_i^{AB} \eta_{jCB} + \eta_j^{AB} \eta_{iCB} = g_{ij} \delta_C^B. \quad (4)$$

At the same time, $\eta_i^{AB} = \sum_{I=0}^{n-1} \frac{1}{2}(\eta_I)_i (\varepsilon_I)^{AB}$. Among the tensor $(\varepsilon_I)^{AB}$ it has only one symmetric tensor $((\varepsilon_0)^{AB} =: \varepsilon^{AB}$ - the metric spin-tensor), the remaining tensors are antisymmetric tensors. Then the structure constants of the orthogonal hypercomplex algebra \mathbb{A} have the form

$$\eta_{ij}^k := \sqrt{2} \eta_i^{AB} \eta_{jCA} \eta_{DB}^k \theta^{CD}, \quad \bar{X}^{A'} = S_A^{A'} X^A, \quad (5)$$

where θ^{CD} ($\theta^{CD} \varepsilon_{CD} := 2$) is an arbitrary symmetric tensor, and $S_A^{A'}$ is generated by the real inclusion $H_i^\Lambda : \mathbb{R}^n \mapsto \mathbb{C}^n$ ($\Lambda, \Psi, \dots = \overline{1, n}$) [2], [3]. Then $e := \frac{1}{\sqrt{2}}\eta^k$ ($\eta^k := (\eta_0)^k$). Therefore,

$$\eta_{ij}^k := \sum_{I=0}^{n-1} \frac{1}{\sqrt{2}} (\eta_I)_i (g_I)_j^k \quad (6)$$

where $g_{ij} =: (g_0)_{ij}$, and the remaining tensors $(g_I)_{ij}$ are antisymmetric tensors. Indeed,

$$\eta_{(ij)}^k := \sqrt{2}(2\eta_{(i}^{AB}\eta_{j)(CA)}\eta_{DB}^k\theta^{CD} - \eta_{(i}^{AB}\eta_{j)AC}\eta_{DB}^k\theta^{CD}) = 2(\frac{1}{\sqrt{2}}\eta_{(i)}\delta_{j)}^k - g_{ij}(\frac{1}{\sqrt{2}}\eta^k) \quad (7)$$

then the axioms 1-3, identity (3a) are executed. Analogically,

$$\begin{aligned} \eta_{(i|j|k)} &= \sqrt{2}\eta_{(i}^{AB}\eta_{j|CA|}\eta_{k)DB}\theta^{CD} = (\frac{1}{\sqrt{2}}\eta_j)g_{ik}, \\ \eta_{i(jk)} &= \sqrt{2}(2\eta_i^{AB}\eta_{(j|CA|}\eta_{k)DB}\theta^{CD} - \eta_i^{AB}\eta_{(j|AC|}\eta_{k)DB}\theta^{CD}) = \\ &= \sqrt{2}(\eta_{(j}g_{k)i} - \eta_{(k}g_{j)i} + \frac{1}{2}\eta_i g_{jk}) = (\frac{1}{\sqrt{2}}\eta_i)g_{jk} \end{aligned} \quad (8)$$

then the identity (3b) is executed,

$$\begin{aligned} \eta_{[ij]}^k(\frac{1}{\sqrt{2}}\eta_k) &= \sqrt{2}\eta_{[i}^{AB}\eta_{j]CA}(\frac{1}{\sqrt{2}}\eta_k)\eta_{DB}^k\theta^{CD} = \eta_{[i}^A\eta_{j]B}\eta_{CA}\theta^{BC} = \\ &= \underbrace{2\eta_{[i}^A\eta_{j]B}\eta_{CA}\theta^{BC}}_{=\eta_{[i}\eta_{j]}=0} - \underbrace{\eta_{[i}^A\eta_{j]B}\eta_{AC}\theta^{(BC)}}_{=0} = 0, \end{aligned} \quad (9)$$

$$< ab - ba, e > = < (a + e)b - b(a + e), a + e > - < ab - ba, a > = 0.$$

then the identity (3b') is executed as a consequence from (3b). Indeed $(\varepsilon^{[A|(BC)|D]} = 0, \varepsilon^{ABCD} := \eta_i^{AB}\eta^{iCD} \quad [2], [3], \theta^{XY} = \theta^{(XY)})$

$$\begin{aligned} \frac{1}{2}\eta_{(i|j}^k\eta_{k|l)}^m &= \eta_{(i}^{AB}\eta_{l)ZC}\eta_{jk}^{CD}\eta_{jXA}\eta_{YB}^m\eta_{TD}^m\theta^{XY}\theta^{ZT} = \\ &= \eta_{(i}^{CD}\eta_{l)ZC}\eta_{jk}^m\eta_{TD}^m\theta^{ZT} - \varepsilon^{ABCD}\eta_{(l|ZC|\eta_i)YB}\eta_{jXA}\eta_{TD}^m\theta^{XY}\theta^{ZT} = \\ &= \eta_{(i}^{CD}\eta_{l)}\eta_{jk}^m\eta_{TD}^m\theta^{ZT}\varepsilon_{ZC} - \frac{1}{2}g_{il}\eta_{jk}^m - \varepsilon^{ABCD}\eta_{(l|ZC|\eta_i)YB}\eta_{jXA}\eta_{TD}^m\theta^{XY}\theta^{ZT}, \\ \frac{1}{2}\eta_{(i|k}^m\eta_{j|l)}^k &= \eta_{(i}^{AB}\eta_{l)ZC}\eta_{jk}^{CD}\eta_{kXA}\eta_{TD}^m\eta_{YB}^m\theta^{XY}\theta^{ZT} = \\ &= \eta_{(i}\eta_{l)ZC}\eta_{jk}^{CD}\eta_{TD}^m\theta^{ZT} - \eta_{(i|TD|\eta_l)ZC}\eta_{jk}^{CD}\eta_{TD}^m + \\ &\quad + \eta_{k}^{BA}\eta_{j}^{CD}\eta_{(i|XA|\eta_l)ZC}\eta_{TD}^m\eta_{YB}^m\theta^{XY}\theta^{ZT} = \\ &= \underbrace{\eta_{(i}\eta_{l)ZC}\eta_{jk}^{CD}\eta_{TD}^m\theta^{ZT}}_1 - \underbrace{\eta_{(i|TD|\eta_l)ZC}\eta_{jk}^{CD}\eta_{TD}^m}_2 + \underbrace{\eta_{j}^{BA}\eta_{(i|XA|\eta_l)}\eta_{YB}^m\theta^{XY}}_1 - \\ &\quad - \varepsilon^{BACD}\eta_{(i|XA|\eta_l)ZC}\eta_{jTD}\eta_{YB}^m\theta^{XY}\theta^{ZT} = \\ &= \underbrace{\eta_{(i}^{CD}\eta_{l)}\eta_{jk}^m\eta_{TD}^m\theta^{ZT}\varepsilon_{ZC}}_1 - \underbrace{\frac{1}{2}g_{il}\eta_{jk}^m}_2 - \varepsilon^{ABCD}\eta_{(l|ZC|\eta_i)YB}\eta_{jTD}\eta_{XA}^m\theta^{ZT}\theta^{XY}. \end{aligned} \quad (10)$$

Otherwise, from (3)

$$\begin{aligned} \frac{1}{2}((ab)a + a(ab)) &= \\ &= -\underbrace{< a, (ab) > e + < a, e > (ab)}_{=<b,e>+<a,a>} + \underbrace{< (ab), e >}_{{-<a,b>+2<b,e>+<a,e>}} a = \\ &= -< a, b > a + < a, e > (ab) + < b, e > \underbrace{(-< a, a > e + 2< a, e > a)}_{a^2} = \\ &= a \underbrace{(-< a, b > e + < a, e > b + < b, e > a)}_{\frac{1}{2}(ab+ba)}, \end{aligned} \quad (10')$$

$$(ab)a = a(ba).$$

There is the alternative-elastic identity

$$((aa)b) - (a(ab)) - (a(ba)) = ((b(aa)) - ((ba)a) - ((ab)a). \quad (11)$$

$$\begin{aligned} ((aa)b) - (a(ab)) - (a(ba)) &= -\langle a, a \rangle b + 2\langle a, e \rangle (ab) - \\ &\quad - 2(a(-\langle a, b \rangle e + \langle a, e \rangle b + \langle b, e \rangle a)) = \\ &= -\langle a, a \rangle b + 2\langle a, b \rangle a - 2\langle b, e \rangle (-\langle a, a \rangle e + 2\langle a, e \rangle a) = \\ &= 2\langle b, e \rangle \langle a, a \rangle e + (2\langle a, b \rangle - 4\langle b, e \rangle \langle a, e \rangle) a - \langle a, a \rangle b, \end{aligned} \quad (12)$$

$$\begin{aligned} (b(aa)) - ((ba)a) - ((ab)a) &= -\langle a, a \rangle b + 2\langle a, e \rangle (ba) - \\ &\quad - 2((- \langle a, b \rangle e + \langle a, e \rangle b + \langle b, e \rangle a)a) = \\ &= -\langle a, a \rangle b + 2\langle a, b \rangle a - 2\langle b, e \rangle (-\langle a, a \rangle e + 2\langle a, e \rangle a) = \\ &= 2\langle b, e \rangle \langle a, a \rangle e + (2\langle a, b \rangle - 4\langle b, e \rangle \langle a, e \rangle) a - \langle a, a \rangle b, \end{aligned}$$

then the axioms 4-5 are executed too. Note that the equations (3) provide the execution of the axiom 3-5, the equations (5), (4) provide the execution of the axiom 1-2,6 and the equations (3). From (3) the common Jordan identity

$$a^k(ba^l) = (a^k b)a^l \quad (13)$$

follows. \square

In addition,

$$g_{kr}\eta_{(i|j|}^k\eta_{l)m}^r = g_{kr}\eta_{j(i}\eta_{|m|l)}^r. \quad (14)$$

This identity follows from (3) and it is called **the weakly normalization identity**. This identity is the normalization identity for $n=8$ only ($g_{kr}\eta_{j(i}\eta_{|m|l)}^r = g_{jm}g_{il}$ in this case). And so this algebra is normalized [2], [3].

Theorem 2. *The metric real numbers, complex numbers, quaternions, octonions, sedenions, hypercomplex Cayley-Dickson numbers possess the identities (3).*

Proof. Let $r = \overline{1, n-1}$ then for the Euclidean metric $\delta_{ij} \forall x$

$$\begin{aligned} x &= x_0 e + \sum_{r=1}^{n-1} i_r x_r, \quad \bar{x} = x_0 e - \sum_{r=1}^{n-1} i_r x_r, \\ i_r i_s &= -i_s i_r, \quad i_r i_r = -e, \quad e i_r = i_r e = i_r, \quad e e = e, \quad \langle i_r, e \rangle = 0. \end{aligned} \quad (15)$$

Let $x := a + bi$, $y := c + di$ where $i := i_{n/2}$ ($r = \overline{1, n/2-1}$) then for the Euclidean metric $\delta_{ij} \forall a, b$

$$a = a_0 e + \sum_{r=1}^{n/2-1} a_r i_r, \quad i e = i, \quad i_r i = i_{r+n/2}, \quad \langle e, a i \rangle = 0. \quad (16)$$

Definition 2. (according to [6, pp.300-303]) 1. Let's define the multiplication for an inductive step according to the Cayley-Dickson double procedure as

$$a(bi) = (ba)i, \quad (ai)b = (a\bar{b})i, \quad (ai)(bi) = -\bar{b}a. \quad (17)$$

2. Let's define the conjugation for an inductive step according to the Cayley-Dickson double procedure as

$$\overline{a + bi} = \bar{a} - bi. \quad (18)$$

Metric hypercomplex Cayley-Dickson algebra possesses the following identities.

1.

Set $\forall a, c : \frac{1}{2}(ac + ca) = \langle a, c \rangle e$ by the induction then the following identity is received

$$\begin{aligned} \frac{1}{2}(x\bar{y} + y\bar{x}) &= \langle x, y \rangle e, \\ (x\bar{y} + y\bar{x}) &= (a + bi)(\bar{c} - di) + (c + di)(\bar{a} - bi) = \\ &= a\bar{c} + \bar{d}b + (bc - da)i + c\bar{a} + \bar{b}d - (bc - da)i = \\ &= 2(\langle a, c \rangle e + \langle d, b \rangle e) = 2\langle x, y \rangle e. \end{aligned} \quad (19)$$

2.

$$\begin{aligned} \frac{1}{2}(yx + xy) &= -\frac{1}{2}(y\bar{x} + x\bar{y}) + \langle x, e \rangle y + \langle y, e \rangle x = \\ &= -\langle x, y \rangle e + \langle x, e \rangle y + \langle y, e \rangle x. \end{aligned} \quad (20)$$

3.

$$2\langle ai, b \rangle = (ai)\bar{b} + b(\overline{ai}) = (ai)\bar{b} - b(ai) = (ab)i - (ab)i = 0. \quad (21)$$

4.

Set $\forall a, c : \overline{ac} = \bar{c}\bar{a}$ by the induction then the following identity is received

$$\begin{aligned} \overline{x\bar{y}} &= \overline{(a + bi)(c + di)} = \overline{ac - \bar{d}b + (b\bar{c} + da)i} = \\ &= (\bar{c}\bar{a} - \bar{b}d) - (b\bar{c} + da)i = \bar{y}\bar{x}. \end{aligned} \quad (22)$$

5.

Set $\forall a, c : \langle ac - ca, e \rangle = 0$ by the induction then the following identity is received

$$\begin{aligned} &\langle xy - yx, e \rangle = \\ &= \langle ((ac - \bar{d}b) + (b\bar{c} + da)i) - ((ca - \bar{b}d) + (bc + d\bar{a}))i, e \rangle = \\ &= \langle (ac - ca) + ((-b + 2\langle b, e \rangle)d - (-d + 2\langle d, e \rangle)b), e \rangle = 0. \end{aligned} \quad (23)$$

6.

Set $\forall a, c : \langle ac, a \rangle = \langle ca, a \rangle = \langle a\bar{c}, a \rangle = \langle \bar{c}a, a \rangle = \langle c, e \rangle \langle a, a \rangle$ by the induction then the following identity is received

$$\begin{aligned} &\langle xy, x \rangle = \langle y, e \rangle \langle x, x \rangle, \\ &\langle xy, x \rangle = \\ &= \langle ac - \bar{d}b + (b\bar{c} + da)i, a + bi \rangle = \\ &= \langle ac, a \rangle + \langle bc, b \rangle + \underbrace{\langle da, b \rangle + \langle db, a \rangle - 2\langle d, e \rangle \langle b, a \rangle}_{= -\langle d(a-b), (a-b) \rangle + \langle d, e \rangle \langle a, a \rangle + \langle d, e \rangle \langle b, b \rangle - 2\langle d, e \rangle \langle b, a \rangle = 0} = \\ &= \langle y, e \rangle \langle x, x \rangle. \end{aligned} \quad (24)$$

□

On the other hand, on the base of the corollary 8.2 [2], [3] it is executed

$$\begin{aligned} \eta_{ij}^k &:= \sum_{I=0}^{n-1} \left(\frac{1}{\sqrt{2}} (\eta_I)^k \right) (-3(h_I)_{ij} + \eta_{(j}(\eta_I)_{i)}) = \\ &= \sqrt{2} \eta_i^{AB} \eta_{jCA} \eta_{DB}^k \frac{2}{N} \varepsilon^{CD} - 3 \sum_{I=1}^{n-1} \left(\frac{1}{\sqrt{2}} (\eta_I)^k \right) (h_I)_{ij} = \\ &= \sqrt{2} (2 - n) \eta_i^{AB} \eta_{jCA} \eta_{DB}^k \frac{2}{N} \varepsilon^{CD} + \sum_{I=1}^{n-1} \left(-3 \left(\frac{1}{\sqrt{2}} (\eta_I)^k \right) (h_I)_{ij} + \sqrt{2} \eta_i^{AB} \eta_{jCA} \eta_{DB}^k \frac{2}{N} \varepsilon^{CD} \right) \end{aligned} \quad (25)$$

where $g_{ij} =: 3(h_0)_{ij}$, and the remaining tensors $(h_I)_{ij}$ are antisymmetric tensors. But $(h_I)_{ij}$ are not arbitrary tensors, there are the compatibility conditions (3)

$$\begin{aligned} \eta_{(i|j|k)} &= (\frac{1}{\sqrt{2}}\eta_j)g_{ik}, & \eta_{i(jk)} &= (\frac{1}{\sqrt{2}}\eta_i)g_{jk}, \\ \sum_{I=1}^{n-1} ((\frac{1}{\sqrt{2}}(\eta_I)_{(k)}(h_{|I|})_{i)j} &= 0, & \sum_{I=1}^{n-1} ((\frac{1}{\sqrt{2}}(\eta_I)_{(k)}(h_{|I|})_{j)i} &= 0. \end{aligned} \quad (26)$$

So the equation (25) takes the form

$$\begin{aligned} \eta_{ij}{}^k &:= \sum_{I=0}^{n-1} (\frac{1}{\sqrt{2}}(\eta_I)^k)(-3(h_I)_{ij} + \eta_{(j}(\eta_I)_{i)}) = \\ &= \sqrt{2}(2-n)\eta_i{}^{AB}\eta_{jCA}\eta^k{}_{DB}\frac{2}{N}\varepsilon^{CD} + \\ &+ \underbrace{\sum_{I=1}^{n-1} ((\frac{1}{\sqrt{2}}(\eta_I)_j)(h_I)_i{}^k - (\frac{1}{\sqrt{2}}(\eta_I)_i)(h_I)_j{}^k - (\frac{1}{\sqrt{2}}(\eta_I)^k)(h_I)_{ij})}_{:= (h_I)_{ij}{}^k} + \sqrt{2}\eta_i{}^{AB}\eta_{jCA}\eta^k{}_{DB}\frac{2}{N}\varepsilon^{CD}. \end{aligned} \quad (27)$$

Note 1. Since for any special (non-special) orthogonal transformation S_i^j according to corollary 8.3 [2], [3], the equation

$$S_i^j \eta_j{}^{AB} = \eta_i{}^{CD} \tilde{S}_C^A \tilde{S}_D^B \quad (S_i^j \eta_j{}^{AB} = \eta_i{}^{DC} \tilde{S}_C^A \tilde{S}_D^B) \quad (28)$$

is executed then any special (non-special) orthogonal transformation S_i^j keeping the algebra identity ($S := \tilde{S} = \tilde{\tilde{S}}$) will transform the structural constants as $S_l^i S_m^j \eta_{ij}{}^k S^r{}_k$ that generates the transformation of the controlling spin-tensor $\theta^{AB} \mapsto \theta^{CD} S_C^A S_D^B$, ($\theta^{AB} \mapsto \frac{4}{N}\varepsilon^{AB} - \theta^{CD} S_C^A S_D^B$) keeping without a change $\eta_i{}^{CD}$ from (5).

Definition 3. Hypercomplex orthogonal algebra \mathbb{A} is called the **homogenous** algebra if the orthogonal transformations S_I exist for all I : $(h_I)_{ij} = \alpha_I (S_I)_i{}^m (h_{gen})_{ml} (S_I)_j{}^l$, $(\eta_I)_i = (S_I)_i{}^m (\eta_{gen})_m$ ($\alpha_I \in \mathbb{R}$, $I = \overline{1, n-1}$).

So, in order to construct an hypercomplex orthogonal homogenous algebra \mathbb{A} , the algebra identity $\frac{1}{\sqrt{2}}\eta^k$ and generator $\frac{1}{\sqrt{2}}(\eta_{gen})^k (h_{gen})_{ij}$ is necessary to know. Then from this generator using orthogonal transformations keeping the algebra identity, $n-1$ basic elements (27) are constructed.

$$\begin{aligned} \eta_{ij}{}^k &= (1 - \underbrace{\sum_{I=1}^{n-1} \alpha_I}_{:= \alpha_0}) \underbrace{\sqrt{2}\eta_i{}^{AB}\eta_{jCA}\eta^k{}_{DB}}_{:= (\eta_0)_{ij}{}^k} \underbrace{\frac{2}{N}\varepsilon^{CD}}_{:= (\theta_0)^{CD}} - \\ &+ \underbrace{\sum_{I=1}^{n-1} \alpha_I ((S_I)_i{}^l (h_{gen})_{lm}{}^r (S_I)^k{}_r (S_I)_j{}^m + \sqrt{2}\eta_i{}^{AB}\eta_{jCA}\eta^k{}_{DB} \frac{2}{N}\varepsilon^{CD})}_{:= (\eta_I)_{ij}{}^k} \end{aligned} \quad (29)$$

Obviously, the equation (29) is a decomposition of the hypercomplex orthogonal homogenous algebra \mathbb{A} ($\eta_{ij}{}^k$) on an algebraic basis \mathbb{A}_I ($(\eta_I)_{ij}{}^k$). In other way, $(\eta_I)_{ij}{}^k := \sqrt{2}\eta_i{}^{AB}\eta_{jCA}\eta^k{}_{DB}(\theta_I)^{CD}$. Define $\theta^{CD} := \sum_{I=0}^{n-1} \alpha_I (\theta_I)^{CD}$ then $\eta_{ij}{}^k$ determine the hyper-complex orthogonal homogenous algebra \mathbb{A} according to (5).

Consider the algorithm 9.1 [2], [3] based on the Bott periodicity [4]:

Algorithm 1. 1. $\Lambda, \dots = \overline{1, n}, i, \dots = \overline{1, n}, A, \dots = \overline{1, 2^{\frac{n}{2}-1}}, \alpha, \dots = \overline{1, n+6}, a, \dots = \overline{1, 2^{\frac{n+6}{2}-1}}$. Suppose there is an orthogonal algebra \mathbb{A} with the structural constants generated from the connection operators η_{Λ}^{AB} with the metric spinor ε^{XZ} and the inclusion operator H_i^{Λ} . We assume that the metric tensor $g_{\Lambda\Psi}$ on the main diagonal contains «+» only. Then we can construct the antisymmetric operators for the space \mathbb{C}^{n+6}

$$\eta_{\alpha}^{ab} = -\eta_{\alpha}^{ba} := \begin{pmatrix} 0 & 0 & 0 & \xi\varepsilon^{AQ} & 0 & \gamma\varepsilon^{AK} & -\alpha\varepsilon^{AD} & \eta_{\Lambda}^A{}_B \\ 0 & 0 & -\xi\varepsilon_{CR} & 0 & -\gamma\varepsilon_{CM} & 0 & (\eta^T)_{\Lambda C}{}^D & \beta\varepsilon_{CB} \\ 0 & \xi\varepsilon_{NY} & 0 & 0 & -\alpha\varepsilon_{NM} & (\eta^T)_{\Lambda N}{}^K & 0 & \delta\varepsilon_{NB} \\ -\xi\varepsilon^{LZ} & 0 & 0 & 0 & \eta_{\Lambda}^L{}_M & \beta\varepsilon^{LK} & -\delta\varepsilon^{LD} & 0 \\ 0 & \gamma\varepsilon_{PY} & \alpha\varepsilon_{PR} & -(\eta^T)_{\Lambda P}{}^Q & 0 & 0 & 0 & -\zeta\varepsilon_{PB} \\ -\gamma\varepsilon^{SZ} & 0 & -\eta_{\Lambda}^S{}_R & -\beta\varepsilon^{SQ} & 0 & 0 & \zeta\varepsilon^{SD} & 0 \\ \alpha\varepsilon^{XZ} & -\eta_{\Lambda}^X{}_Y & 0 & \delta\varepsilon^{XQ} & 0 & -\zeta\varepsilon^{XK} & 0 & 0 \\ -(\eta^T)_{\Lambda T}{}^Z & -\beta\varepsilon_{TY} & -\delta\varepsilon_{TR} & 0 & \zeta\varepsilon_{TM} & 0 & 0 & 0 \end{pmatrix}, \quad (30)$$

$$\eta_{\alpha ab} = -\eta_{\alpha ba} := \begin{pmatrix} 0 & 0 & 0 & -\zeta\varepsilon_{AQ} & 0 & -\delta\varepsilon_{AK} & -\beta\varepsilon_{AD} & \eta_{\Lambda A}{}^B \\ 0 & 0 & \zeta\varepsilon^{CR} & 0 & \delta\varepsilon^{CM} & 0 & (\eta^T)_{\Lambda}{}^C{}_D & \alpha\varepsilon^{CB} \\ 0 & -\zeta\varepsilon^{NY} & 0 & 0 & -\beta\varepsilon^{NM} & (\eta^T)_{\Lambda}{}^N{}_K & 0 & -\gamma\varepsilon^{NB} \\ \zeta\varepsilon_{LZ} & 0 & 0 & 0 & \eta_{\Lambda L}{}^M & \alpha\varepsilon_{LK} & \gamma\varepsilon_{LD} & 0 \\ 0 & -\delta\varepsilon^{PY} & \beta\varepsilon^{PR} & -(\eta^T)_{\Lambda}{}^P{}_Q & 0 & 0 & 0 & \xi\varepsilon^{PB} \\ \delta\varepsilon_{SZ} & 0 & -\eta_{\Lambda S}{}^R & -\alpha\varepsilon_{SQ} & 0 & 0 & -\xi\varepsilon_{SD} & 0 \\ \beta\varepsilon_{XZ} & -\eta_{\Lambda X}{}^Y & 0 & -\gamma\varepsilon_{XQ} & 0 & \xi\varepsilon_{XK} & 0 & 0 \\ -(\eta^T)_{\Lambda}{}^T{}_Z & -\alpha\varepsilon^{TY} & \gamma\varepsilon^{TR} & 0 & -\xi\varepsilon^{TM} & 0 & 0 & 0 \end{pmatrix}, \quad (30')$$

$$\begin{aligned} \alpha &:= \frac{1}{2}(i\eta_{n+1} + \eta_{n+2}), & \beta &:= \frac{1}{2}(-i\eta_{n+1} + \eta_{n+2}), \\ \gamma &:= \frac{1}{2}(\eta_{n+3} + i\eta_{n+4}), & \delta &:= \frac{1}{2}(-\eta_{n+3} + i\eta_{n+4}), \\ \xi &:= \frac{1}{2}(\eta_{n+5} + i\eta_{n+6}), & \zeta &:= \frac{1}{2}(-\eta_{n+5} + i\eta_{n+6}). \end{aligned} \quad (31)$$

2. $\Lambda, \dots = \overline{1, n+8}, i, \dots = \overline{1, n+8}, A, \dots = \overline{1, 2^{\frac{n+8}{2}-1}}, \alpha, \dots = \overline{1, n+6}, a, \dots = \overline{1, 2^{\frac{n+6}{2}-1}}$. Transition to the connection operators of the space \mathbb{C}^{n+8} is carried out as follows:

$$\eta_{\Lambda}^{AB} := \begin{pmatrix} \eta_{\alpha}^{ab} & \phi\delta_a^d \\ \psi\delta_c^b & -(\eta^T)_{\alpha cd} \end{pmatrix}, \quad (32)$$

$$\phi := \frac{1}{2}(i\eta_{n+7} + \eta_{n+8}), \quad \psi := \frac{1}{2}(-i\eta_{n+7} + \eta_{n+8}) \quad (33)$$

with the metric spinor $\varepsilon^{XZ} := \begin{pmatrix} 0 & \delta_a^d \\ \delta_c^b & 0 \end{pmatrix}$, $\varepsilon_{XZ} := \begin{pmatrix} 0 & \delta_a^d \\ \delta_c^b & 0 \end{pmatrix}$. Then we go to the connection operators of the space $\mathbb{R}^{n+8} \subset \mathbb{C}^{n+8}$ using the corresponding inclusion operator. And such the operators generate the structure constants of the sedenion algebra with dimension equal to $n+8$.

Note 2. In the conditions of the algorithm 1 and the examples 1, 2 the algebra identity is $\frac{1}{\sqrt{2}}\eta_{n+8}$ (or accordingly $\frac{1}{\sqrt{2}}\eta_n$). Therefore, for reduction of designations in conformity, it is necessary to make the redesignation: $n+8 \mapsto 0$ (or accordingly $n \mapsto 0$).

Table 2: The multiplication table example of the algebra $\mathbb{A}_{gen} ((\eta_{gen})_{ij}^k)$.

*	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	\dots	e_{n+1}	e_{n+2}	e_{n+3}	e_{n+4}	e_{n+5}	e_{n+6}	e_{n+7}	e_{n+8}
e_1	$-e_{n+8}$	e_{n+7}							\dots							$-e_2$	e_1
e_2	$-e_{n+7}$	$-e_{n+8}$							\dots							e_1	e_2
e_3			$-e_{n+8}$	e_{n+7}					\dots							$-e_4$	e_3
e_4			$-e_{n+7}$	$-e_{n+8}$					\dots							e_3	e_4
e_5					$-e_{n+8}$	e_{n+7}			\dots							$-e_6$	e_5
e_6					$-e_{n+7}$	$-e_{n+8}$			\dots							e_5	e_6
e_7							$-e_{n+8}$	$-e_{n+7}$	\dots							e_8	e_7
e_8							e_{n+7}	$-e_{n+8}$	\dots							$-e_7$	e_8
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
e_{n+1}									\dots	$-e_{n+8}$	$-e_{n+7}$					e_{n+2}	e_{n+1}
e_{n+2}									\dots	e_{n+7}	$-e_{n+8}$					$-e_{n+1}$	e_{n+2}
e_{n+3}									\dots			$-e_{n+8}$	e_{n+7}			$-e_{n+4}$	e_{n+3}
e_{n+4}									\dots			$-e_{n+7}$	$-e_{n+8}$			e_{n+3}	e_{n+4}
e_{n+5}									\dots					$-e_{n+8}$	e_{n+7}	$-e_{n+6}$	e_{n+5}
e_{n+6}									\dots					$-e_{n+7}$	$-e_{n+8}$	e_{n+5}	e_{n+6}
e_{n+7}	e_2	$-e_1$	e_4	$-e_3$	e_6	$-e_5$	$-e_8$	e_7	\dots	$-e_{n+2}$	e_{n+1}	e_{n+4}	$-e_{n+3}$	e_{n+6}	$-e_{n+5}$	$-e_{n+8}$	e_{n+7}
e_{n+8}	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	\dots	e_{n+1}	e_{n+2}	e_{n+3}	e_{n+4}	e_{n+5}	e_{n+6}	e_{n+7}	e_{n+8}

Note 3. The controlling spinor $X^A = (1, 0, 0, 0, 1, 0, 0, 0)$ from (37) generates the oktonion algebra entirely. But tensor

$$h_{ij} := H_i^A H_j^\Psi (\eta_\Lambda^A {}_B Y_A \eta_{\Psi C}^B X^C), \quad X^C = (1, 0, 0, 0, 0, 0, 0, 0), \quad Y_A = (1, 0, 0, 0, 0, 0, 0, 0) \quad (42)$$

is generated by the equation (39) as usual. Therefore, $n = 8$ is the initial induction step.

The last paragraph of the example 1 should be clarified. Let the hypercomplex orthogonal algebra is given by the structural constant η_{ij}^k . Then these constants can be expanded according to (3) as

$$\eta_{ij}^k := (\eta_0)_{ij}^k + (\eta_a)_{ij}^k. \quad (43)$$

From (29) and corollary 8.2 [2], [3]

$$(\eta_0)_{ij}^k := (\frac{1}{\sqrt{2}}\eta_i)\delta_j^k + (\frac{1}{\sqrt{2}}\eta_j)\delta_i^k - g_{ij}(\frac{1}{\sqrt{2}}\eta^k) = \sqrt{2}\eta_i^{AB}\eta_{jCA}\eta_{DB}^k \underbrace{\frac{2}{N}\varepsilon^{CD}}_{(\theta_0)^{CD}},$$

$$\eta_{(ij)k} = (\eta_0)_{(ij)k}, \quad \eta_{[i|j|k]} = (\eta_0)_{[i|j|k]}, \quad \eta_{i(jk)} = (\eta_0)_{i(jk)}, \quad (44)$$

$$\begin{cases} (\eta_a)_{(ij)k} = 0, \\ (\eta_a)_{[i|j|k]} = 0, \\ (\eta_a)_{i(jk)} = 0, \end{cases} \Rightarrow (\eta_a)_{ijk} = \eta_{[ijk]}, \quad (\frac{1}{\sqrt{2}}\eta^i)(\eta_a)_{ijk} = 0.$$

Define

$$(\eta_a)_{ij}^k := \sqrt{2}\eta_i^{AB}\eta_{jCA}\eta_{DB}^k (\theta_a)^{CD} = -\sqrt{2}\eta_i^{AB}\eta_{jAC}\eta_{DB}^k (\theta_a)^{CD}. \quad (45)$$

This equation always has the particular solution

$$(\theta_a)^{CD} = (\theta_a)^{DC} := -\frac{4}{3\sqrt{2}N}(\eta_a)_{lm}{}^r \eta^l{}_{XY} \eta^{mXC} \eta_r{}^{DY} = \frac{4}{3\sqrt{2}N}(\eta_a)_{lm}{}^r \eta^l{}_{XY} \eta^{mCX} \eta_r{}^{DY}. \quad (46)$$

This statement follows from the equation executed for all even $n \geq 8$ (note 16.1 [2], [3])

$$\eta_{[i}^{AB} \eta_{j]AC} \eta^{[m|XC|} \eta^{l]}_{XY} \eta_r^{DY} \eta^k_{DB} = \frac{N}{2} (\delta_r^{[l} \delta_{[j}^{m]} \delta_i^{k]} - g^{k[l} \delta_{[j}^{m]} g_{i]r}) + \frac{N}{4} \delta_r^k \delta_i^{[l} \delta_j^{m]}. \quad (47)$$

And this identity is a consequence of the identities (16.28) and (16.31) [2], [3]. Thus,

$$\theta^{CD} := (\theta_0)^{CD} + (\theta_a)^{CD} \quad (48)$$

that proves the theorem.

Theorem 3. *Every hypercomplex orthogonal algebra \mathbb{A} admits the decomposition (5).*

Note 4. *The orthogonal transformations S_i^j from the group $O(n, \mathbb{R})$ ($SO(n, \mathbb{R})$) generate the pinor (spinor) transformations S_A^B from the group $pin(n, \mathbb{R})$ ($Spin(n, \mathbb{R})$) which are allocated with the real structure by the involution $S_A^{B'}$ according to (6.41) from [2], [3]. The pinor (spinor) transformations represent the subgroup of the orthogonal group $O_{\mathbb{R}}(2^{\frac{n}{2}-1}, \mathbb{C})$ ($SO_{\mathbb{R}}(2^{\frac{n}{2}-1}, \mathbb{C})$) (in the sense $S_A^B S_C^D \varepsilon_{BD} = \varepsilon_{AC}$, $\bar{S}_{A'}^{B'} = \bar{S}_{A'}^C S_C^D S_D^{B'}$ where ε_{BD} is the metric spinor). The orthogonal transformations from the group $O(n, \mathbb{R})$ ($SO(n, \mathbb{R})$) keeping the algebra identity cause the transformations of the controlling spinor θ^{CD} without changing the connection operators η_i^{AB} . Therefore, the quotient group $O_{\mathbb{R}}(2^{\frac{n}{2}-1}, \mathbb{C})/pin(n, \mathbb{R})$ ($SO_{\mathbb{R}}(2^{\frac{n}{2}-1}, \mathbb{C})/Spin(n, \mathbb{R})$) will implement the classification of such the hypercomplex orthogonal homogenous algebras \mathbb{A} . Besides, the classification is carried out on own values of the controlling spinor θ^{CD} (48) because any symmetric spinor θ^{CD} (48) is led to a diagonal form by the orthogonal transformation from the group $SO_{\mathbb{R}}(2^{\frac{n}{2}-1}, \mathbb{C})$.*

Theorem 4. *Hypercomplex metric Cayley-Dickson algebra is the hypercomplex special orthogonal homogenous algebra \mathbb{A} .*

Proof. Let $x := a + bi$ where $i := i_{n/2}$. Set that any hypercomplex metric Cayley-Dickson algebra $\mathbb{A}^{\frac{n}{2}}$ is the hypercomplex special orthogonal homogenous algebra by the induction. Then it has $(h_{gen})_{ij} := (h_1)_{ij}$ and $(h_I)_{ij} = \alpha_I (S_I)_i^m (h_{gen})_{ml} (S_I)_j^l$ ($\alpha_I \in \mathbb{R}$, $I = \overline{1, \frac{n}{2} - 1}$, $i, j, m, l = \overline{0, \frac{n}{2} - 1}$, $a, b, c = \overline{\frac{n}{2}, n - 1}$, $\alpha, \beta = \overline{0, n - 1}$). Let the h_{gen} has the form $(h_{ab} := \delta_a^i \delta_b^j h_{ij}$, $h_{aj} := \delta_a^i h_{ij}$, $h_{ib} := \delta_b^j h_{ij}$, $\delta_a^i : i_{1+n/2} \rightarrow i_I$)

$$(h_{gen})_{\alpha\beta} = \begin{pmatrix} (h_1)_{ij} & 0 \\ 0 & -(h_1)_{ab} + \frac{1}{2}((\eta_{n/2})_a(\eta_{1+n/2})_b - (\eta_{1+n/2})_a(\eta_{n/2})_b) \end{pmatrix}. \quad (49)$$

Then for the hypercomplex metric Cayley-Dickson algebra \mathbb{A}^n exists three types of the basic elements only.

1.

$$(h_I)_{\alpha\beta} = \begin{pmatrix} (h_I)_{ij} & 0 \\ 0 & -(h_I)_{ab} + \frac{1}{2}((\eta_{n/2})_a(\eta_{I+n/2})_b - (\eta_{I+n/2})_a(\eta_{n/2})_b) \end{pmatrix}. \quad (50)$$

In this case the special orthogonal transformations $(S_I)_i^m$ leave motionless the identity vector $e = \frac{1}{\sqrt{2}} \eta_0$. Hence, analogical transformations $(S_I)_a^b$ leave motionless

the vector $i = ei = \frac{1}{\sqrt{2}}\eta_{n/2}$. Thus, the special orthogonal transformations have the form

$$(S_I)_\alpha^\beta := \begin{pmatrix} (S_I)_m^i & 0 \\ 0 & (S_I)_c^a \end{pmatrix}. \quad (51)$$

2.

$$\begin{pmatrix} 0 & -(h_I)_{ia} + \frac{1}{2}(\eta_I)_i(\eta_{n/2})_a \\ -(h_I)_{bj} - \frac{1}{2}(\eta_{n/2})_b(\eta_I)_j & 0 \end{pmatrix}. \quad (52)$$

In this case the special orthogonal transformations leave motionless the identity vector and the vector rail line i with changing the direction, convert the vector i_1 to the vector $i_{I+n/2}$ and vice versa, and have the form $(S_{I+\frac{n}{2}})_\alpha^\beta := (\tilde{S}_I)_\alpha^\gamma (S_I)_\gamma^\beta$ where

$$(\tilde{S}_I)_\alpha^\beta := \begin{pmatrix} -\frac{1}{\sqrt{2}}\delta_i^k + \frac{1}{2\sqrt{2}}(\eta_I)_i(\eta_I)^k + \frac{1}{2}(1 + \frac{1}{\sqrt{2}})(\eta_0)_i(\eta_0)^k & \\ \frac{1}{\sqrt{2}}\delta_b^k + \frac{1}{2}(1 - \frac{1}{\sqrt{2}})(\eta_{I+n/2})_b(\eta_I)^k - \frac{1}{2\sqrt{2}}(\eta_{n/2})_b(\eta_0)^k & \\ \frac{1}{\sqrt{2}}\delta_i^a + \frac{1}{2}(1 - \frac{1}{\sqrt{2}})(\eta_I)_i(\eta_{I+n/2})^a - \frac{1}{2\sqrt{2}}(\eta_0)_i(\eta_{n/2})^a & \\ \frac{1}{\sqrt{2}}\delta_b^a - \frac{1}{2\sqrt{2}}(\eta_{I+n/2})_b(\eta_{I+n/2})^a - \frac{1}{2}(1 + \frac{1}{\sqrt{2}})(\eta_{n/2})_b(\eta_{n/2})^a & \end{pmatrix}. \quad (53)$$

3.

In this case the special orthogonal transformation $S_{n/2}$ leave motionless all the vectors with $\begin{cases} \text{even index } r, & r < n/2; \\ \text{odd index } r, & r \geq n/2. \end{cases}$ Then the remaining transformations is such ($\langle i_1, i_1 \rangle = -1$):

$$\begin{cases} -i_1 \rightarrow i_{n/2}, & r = 1; \\ i_{n/2} \rightarrow i_1, & r = n/2; \\ \langle i_r i_{r-1}, i_1 \rangle i_r \rightarrow i_{n/2+r-1}, & r = 2s + 1 < n/2, \quad r > 1; \\ \langle i_{r+1} i_r, i_1 \rangle i_r \rightarrow i_{r-n/2+1}, & r = 2s > n/2. \end{cases} \quad (54)$$

□

Thus, all is made necessary for technical realization of the algorithm 1.

Example 2. Let $n=16$. The algorithm 1 is realized in the Appendix.

1. This article contains the file "sedenion.pas" (by the operator \input{sedenion.pas}) being a programming unit adapted to the LaTeX (LaTeX version this article on <http://arxiv.org/>) for the Delphi. At the same time, this file is the Appendix to this article. You must create a project with this "unit sedenion" and put on the form Button1: TButton, StringGrid1: TStringGrid (the lines 22-24).
2. At error occurrence "Stack overflow" it is necessary to adjust the line 15.
3. At the lines 146-163 the connecting operators $\sqrt{2}\eta_i^{AB}$ for \mathbb{R}^8 ($i, A, B = \overline{1,8}$) is constructed.
4. At the lines 164-166 the metric spinor ε_{AB} is constructed.
5. At the lines 167-183 the connecting operators $\sqrt{2}\eta_i^A{}_B$, $\sqrt{2}\eta_{iA}{}^B$, $\sqrt{2}(\eta^T)_i^A{}_B$, $\sqrt{2}(\eta^T)_{iA}{}^B$ is constructed.

6. At the lines 186-251 the connecting operators multiplied by $\sqrt{2}$ for $n=14$ is constructed according to the step 1 of the algorithm 1.
7. At the lines 252-289 the connecting operators for $n=16$ is constructed according to the step 2 of the algorithm 1.
8. At the lines 290-294 the controlling spinors X^A is constructed according to (34).
9. At the lines 295-303 the inclusion operators $P_j^A := \eta_{jAB} X^A$ is constructed according to (38).
10. At the lines 304-333 the structural constants $(\eta_{gen})_{ij}^k := \sqrt{2}\eta_i^{AB} P_{iA} P_{kB}$ is constructed according to (5),(35).
11. At the lines 334-479 the basic orthogonal transformation S_I is constructed according to (29).
12. At the lines 480-534 the canonical sedenion structural constants is constructed according to (29).
13. At the lines 535-548 the canonical sedenion structural constants is outputted.
14. At the lines 549-584 the canonical sedenion structural constants is outputted into the file:

Table 3: The canonical sedenion multiplication table.

*	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	$-e_0$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	$-e_0$	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	$-e_0$	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_{11}	e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	$-e_0$	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	$-e_0$	$-e_1$	$-e_2$	$-e_3$
e_{13}	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	$-e_0$	e_3	$-e_2$
e_{14}	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	$-e_0$	e_1
e_{15}	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	$-e_0$

15. Algorithm 1 is realized by the scheme "one-to-one" and therefore, it requires more memory allocation (the line 15). In order to apply it to higher dimensions, it should be optimized.

16. *System characteristics of the computer on which the program is tested:*
HP Pavilion dv7-6b50er i3-2330M/4096/500/Radeon HD6770 2Gb/Win7 HP64
17. *Run time <3c.*

Appendix.

```

1 //The sedenion multiplication table.
2 { ***** }
3 { }
4 { Sedenion Unit for Delphi }
5 { }
6 { Copyright (c) 2011 Konstantin Andreev All Rights Reserved }
7 { THIS SOFTWARE IS PROVIDED "AS IS" WITHOUT WARRANTY OF ANY KIND, EITHER EXPRESSED OR IMPLIED, }
8 { INCLUDING BUT NOT LIMITED TO THE IMPLIED MERCHANTABILITY AND/OR FITNESS FOR A PARTICULAR }
9 { PURPOSE. KONSTANTIN ANDREEV CANNOT BE HELD RESPONSIBLE FOR ANY LOSSES, EITHER DIRECT OR }
10 { INDIRECT, OF ANY PARTY MAKING USE OF THIS SOFTWARE. IN MAKING USE OF THIS SOFTWARE, YOU }
11 { AGREE TO BE BOUND BY THE TERMS AND CONDITIONS FOUND IN THE ACCOMPANYING LICENSE. }
12 { ***** }
13 { }
14 unit sedenion;
15 {$M 16384,9000000}
16 { }
17 interface
18 uses
19   Windows, Messages, SysUtils, Classes, Graphics, Controls, Forms, Dialogs,
20   StdCtrls, Grids, IniFiles;
21 type
22   TForm1 = class(TForm)
23     Button1: TButton;
24     StringGrid1: TStringGrid;
25     procedure Button1Click(Sender: TObject);
26   end;
27 var
28   Form1: TForm1;
29 implementation
30 {$R *.DFM} //
31
32 procedure TForm1.Button1Click(Sender: TObject);
33 type
34   complex = record   x,y      : real; end;
35 var
36   eta_16      : array [1..16,1..128,1..128] of complex; //The connection operators
37   eta_16_      : array [1..16,1..128,1..128] of complex; //for n=16  $\eta_i^{AB}$ .
38   eta_8        : array [1..8,1..8,1..8] of complex; //The connection operators
39   eta_8_       : array [1..8,1..8,1..8] of complex; //for n=16  $\eta_i^{AB}$ .
40   eta_8__      : array [1..8,1..8,1..8] of complex; //The connection operators
41   eta_8_T      : array [1..8,1..8,1..8] of complex; //for n=8  $\eta_i^{AB}$ .
42   eta_8__T     : array [1..8,1..8,1..8] of complex; //The connection operators
43   eta          : array [1..16,1..16,1..16] of complex; //contracted with the metric
44   eta_         : array [1..16,1..128,1..16] of complex; //spinor for n=8  $\eta_i^{AB}$ .
45   e_8          : array [1..8,1..8] of complex; //The connection operators
46   P            : array [1..16,1..128] of complex; //contracted with the metric
47   x            : array [1..128] of complex; //spinor for n=8  $\eta_i^{AB}$ .
48   unit_        : array [0..15] of complex; //The connection operators
49   g            : array [0..15,0..15] of complex; //contracted with the metric
50   i,j,k,l,m,r  : integer; //spinor for n=8  $(\eta^T)_i^{AB}$ .
51   mi           : complex; //The connection operators
52   m_i          : complex; //contracted with the metric
53   m_12         : complex; //spinor for n=8  $(\eta^T)_{iA}^B$ .
54   m_3          : complex; //The connection operators
55   IniFile      : TIniFile; //contracted with the metric
56   array_str    : string; //spinor for n=8  $(\eta^T)_i^{AB}$ .
57   s_orthogonal : array [1..15,0..15,1..16] of complex; //The connection operators
58   eta_orthogonal : array [1..15,0..16,0..16,0..16] of complex; //contracted with the metric
59   eta_orthogonal_ : array [1..15,0..16,0..16,0..16] of complex; //spinor for n=16  $(\eta_{gen})_{ij}^k$  from (41).
60   eta_constant : array [0..15,0..15,0..15] of complex; //The auxiliary variable
61
62   //The metric of the Euclidean
63   //space  $\mathbb{R}^{16}$ 
64   //  $g_{ij} = g^{ij} = \delta_i^j$ .
65   //Indices of an array element.
66   //The complex factor  $i$ .
67   //The complex factor  $-i$ .
68   //The complex factor  $-1$ .
69   //The complex factor  $-12$ .
70   //The complex factor  $\frac{1}{3}$ .
71   //The output file.
72   //A line of the output file.
73   //The basic orthogonal
74   //transformations
75   //for n=16  $(S_I)_i^j$  ( $I = \overline{1,15}$ ) from note 2.
76   //The structural constant
77   //of hypercomplex basic
78   //algebras for n=16  $(\eta)_{ij}^k$ .
79   //The auxiliary variable.
80   //The structural constant
81   //of the canonical sedenion
82   //algebra  $\eta_{ij}^k$ .

```

```

88 //Addition of complex numbers.
89 function add(c11,c12: complex):complex;
90 var c13 : complex;
91 begin
92   c13.x:=c11.x+c12.x;
93   c13.y:=c11.y+c12.y;
94   add:=c13;
95 end;
96 //Multiplication of complex numbers.
97 function mul(c21,c22: complex):complex;
98 var c13 : complex;
99 begin
100   c13.x:=c21.x*c22.x-c21.y*c22.y;
101   c13.y:=c21.x*c22.y+c21.y*c22.x;
102   mul:=c13;
103 end;
104 //Initialization of a complex number.
105 procedure Init_(var c31 : complex);
106 begin
107   c31.x:=0; c31.y:=0;
108 end;
109 begin
110 //Initialization the connection operators for n=8.
111   for i:=1 to 8 do
112     for j:=1 to 8 do
113       for k:=1 to 8 do
114         begin
115           Init_(eta_8[i,j,k]);
116           Init_(eta_8_[i,j,k]);
117         end;
118 //Initialization the metric spinor for n=8.
119       for i:=1 to 8 do
120         for j:=1 to 8 do
121           Init_(e_8[i,j]);
122 //Initialization the connection operators for n = 16.
123         for i:=1 to 16 do
124           for j:=1 to 128 do
125             for k:=1 to 128 do
126               begin
127                 Init_(eta_16[i,j,k]);
128                 Init_(eta_16_[i,j,k]);
129               end;
130 //Initialization the ortogonal transformation for n = 16.
131             for m:=1 to 15 do
132               for i:=1 to 16 do
133                 for j:=1 to 16 do
134                   Init_(s_orthogonal[m,i,j]);
135 //Initialization the sedenion algebra identity for n=16.
136                 for i:=0 to 15 do
137                   Init_(unit_[i]);
138                   unit_[0].x:=1;
139 //Initialization the metric tensor.
140                 for i:=0 to 15 do
141                   begin
142                     for j:=0 to 15 do
143                       Init_(g[i,j]);
144                       g[i,i].x:=1;
145                     end;
146 //Construction the connection operators (generating the octonion algebra) for n=8 (multiplied by  $\sqrt{2}$ ).
147                     eta_8[1,1,2].y:=+1; eta_8[1,3,4].y:=+1; eta_8[1,5,6].y:=-1; eta_8[1,7,8].y:=-1;
148                     eta_8[2,1,2].x:=-1; eta_8[2,3,4].x:=+1; eta_8[2,5,6].x:=-1; eta_8[2,7,8].x:=+1;
149                     eta_8[3,1,3].y:=-1; eta_8[3,2,4].y:=+1; eta_8[3,5,7].y:=+1; eta_8[3,6,8].y:=-1;
150                     eta_8[4,1,3].x:=+1; eta_8[4,2,4].x:=+1; eta_8[4,5,7].x:=+1; eta_8[4,6,8].x:=+1;
151                     eta_8[5,1,4].y:=+1; eta_8[5,2,3].y:=+1; eta_8[5,5,8].y:=-1; eta_8[5,6,7].y:=-1;
152                     eta_8[6,1,4].x:=-1; eta_8[6,2,3].x:=+1; eta_8[6,5,8].x:=-1; eta_8[6,6,7].x:=+1;
153                     for i:=1 to 8 do
154                       for j:=1 to 8 do
155                         for k:=1 to j do
156                           begin
157                             eta_8[i,j,k].x:=-eta_8[i,k,j].x;
158                             eta_8[i,j,k].y:=-eta_8[i,k,j].y;
159                           end;
160                         eta_8[7,1,5].y:=+1; eta_8[7,2,6].y:=+1; eta_8[7,3,7].y:=+1; eta_8[7,4,8].y:=+1;
161                         eta_8[7,5,1].y:=-1; eta_8[7,6,2].y:=-1; eta_8[7,7,3].y:=-1; eta_8[7,8,4].y:=-1;
162                         eta_8[8,1,5].x:=+1; eta_8[8,2,6].x:=+1; eta_8[8,3,7].x:=+1; eta_8[8,4,8].x:=+1;
163                         eta_8[8,5,1].x:=+1; eta_8[8,6,2].x:=+1; eta_8[8,7,3].x:=+1; eta_8[8,8,4].x:=+1;
164 //Construction the metric spinor for n = 8.
165                         e_8[1,5].x:=1; e_8[2,6].x:=1; e_8[3,7].x:=1; e_8[4,8].x:=1;
166                         e_8[5,1].x:=1; e_8[6,2].x:=1; e_8[7,3].x:=1; e_8[8,4].x:=1;
167 //Construction the connection operators contracted with the metric spinor for n=8 (multiplied by  $\sqrt{2}$ ).
168                       for i:=1 to 8 do
169                         for j:=1 to 8 do
170                           for k:=1 to 8 do
171                             begin
172                               Init_(eta_8_[i,j,k]);
173                               Init_(eta_8__T[i,j,k]);
174                               Init_(eta_8__T[i,j,k]);
175                               Init_(eta_8__T[i,j,k]);
176                               for m:=1 to 8 do
177                                 begin
178                                   eta_8__[i,j,k]:=add(eta_8_[i,j,k],mul(eta_8[i,j,m],e_8[m,k]));
179                                   eta_8__T[i,j,k]:=add(eta_8__T[i,j,k],mul(eta_8[i,m,j],e_8[m,k]));
180                                   eta_8__[i,j,k]:=add(eta_8__[i,j,k],mul(eta_8[i,m,k],e_8[m,j]));
181                                   eta_8__T[i,j,k]:=add(eta_8__T[i,j,k],mul(eta_8[i,k,m],e_8[m,j]));
182                                 end;
183                               end;

```

```

184 //The constant factors:  $i(mi)$ ,  $-i(m_-i)$ ,  $-1(m_-)$ .
185 mi.x:=0; mi.y:=1; m_i.x:=0; m_i.y:=-1; m_.x:=-1; m_.y:=0;
186 //Construction the connection operators for n=14 from (30),(30') (multiplied by  $\sqrt{2}$ ).
187 //1. n=13,14.
188 for i:=1 to 8 do
189   for l:=1 to 8 do
190     begin
191       eta_16[13,i+ 0,l+24]:=e_8[i,l]; eta_16[14,i+ 0,l+24]:=mul(mi,e_8[i,l]);
192       eta_16[13,i+ 8,l+16]:=mul(m_,e_8[i,l]); eta_16[14,i+ 8,l+16]:=mul(m_i,e_8[i,l]);
193
194       eta_16[13,i+32,l+56]:=e_8[i,l]; eta_16[14,i+32,l+56]:=mul(m_i,e_8[i,l]);
195       eta_16[13,i+40,l+48]:=mul(m_,e_8[i,l]); eta_16[14,i+40,l+48]:=mul(m_i,e_8[i,l]);
196
197       eta_16[13,i+ 0,l+24]:=e_8[i,l]; eta_16[14,i+ 0,l+24]:=mul(m_i,e_8[i,l]);
198       eta_16[13,i+ 8,l+16]:=mul(m_,e_8[i,l]); eta_16[14,i+ 8,l+16]:=mul(m_i,e_8[i,l]);
199
200       eta_16[13,i+32,l+56]:=e_8[i,l]; eta_16[14,i+32,l+56]:=mul(mi,e_8[i,l]);
201       eta_16[13,i+40,l+48]:=mul(m_,e_8[i,l]); eta_16[14,i+40,l+48]:=mul(m_i,e_8[i,l]);
202     end;
203 //2. n=11,12.
204 for i:=1 to 8 do
205   for l:=1 to 8 do
206     begin
207       eta_16[11,i+ 0,l+40]:=e_8[i,l]; eta_16[12,i+ 0,l+40]:=mul(mi,e_8[i,l]);
208       eta_16[11,i+ 8,l+32]:=mul(m_,e_8[i,l]); eta_16[12,i+ 8,l+32]:=mul(m_i,e_8[i,l]);
209
210       eta_16[11,i+16,l+56]:=mul(m_,e_8[i,l]); eta_16[12,i+16,l+56]:=mul(mi,e_8[i,l]);
211       eta_16[11,i+24,l+48]:=e_8[i,l]; eta_16[12,i+24,l+48]:=mul(m_i,e_8[i,l]);
212
213       eta_16[11,i+ 0,l+40]:=e_8[i,l]; eta_16[12,i+ 0,l+40]:=mul(m_i,e_8[i,l]);
214       eta_16[11,i+ 8,l+32]:=mul(m_,e_8[i,l]); eta_16[12,i+ 8,l+32]:=mul(m_i,e_8[i,l]);
215
216       eta_16[11,i+16,l+56]:=mul(m_,e_8[i,l]); eta_16[12,i+16,l+56]:=mul(m_i,e_8[i,l]);
217       eta_16[11,i+24,l+48]:=e_8[i,l]; eta_16[12,i+24,l+48]:=mul(m_i,e_8[i,l]);
218     end;
219 //3. n=9,10.
220 for i:=1 to 8 do
221   for l:=1 to 8 do
222     begin
223       eta_16[9,i+ 0,l+48]:=mul(m_i,e_8[i,l]); eta_16[10,i+ 0,l+48]:=mul(m_,e_8[i,l]);
224       eta_16[9,i+ 8,l+56]:=mul(m_i,e_8[i,l]); eta_16[10,i+ 8,l+56]:=e_8[i,l];
225
226       eta_16[9,i+16,l+32]:=mul(m_i,e_8[i,l]); eta_16[10,i+16,l+32]:=mul(m_,e_8[i,l]);
227       eta_16[9,i+24,l+40]:=mul(m_i,e_8[i,l]); eta_16[10,i+24,l+40]:=e_8[i,l];
228
229       eta_16[9,i+ 0,l+48]:=mul(mi,e_8[i,l]); eta_16[10,i+ 0,l+48]:=mul(m_,e_8[i,l]);
230       eta_16[9,i+ 8,l+56]:=mul(mi,e_8[i,l]); eta_16[10,i+ 8,l+56]:=e_8[i,l];
231
232       eta_16[9,i+16,l+32]:=mul(mi,e_8[i,l]); eta_16[10,i+16,l+32]:=mul(m_,e_8[i,l]);
233       eta_16[9,i+24,l+40]:=mul(mi,e_8[i,l]); eta_16[10,i+24,l+40]:=e_8[i,l];
234     end;
235 //4. n=1-8.
236 for i:=1 to 8 do
237   for l:=1 to 8 do
238     for k:=1 to 8 do
239       begin
240         eta_16[k,i+ 0,l+56]:=eta_8[k,i,l];
241         eta_16[k,i+ 8,l+48]:=eta_8_T[k,i,l];
242
243         eta_16[k,i+16,l+40]:=eta_8_T[k,i,l];
244         eta_16[k,i+24,l+32]:=eta_8[k,i,l];
245
246         eta_16[k,i+ 0,l+56]:=eta_8[k,i,l];
247         eta_16[k,i+ 8,l+48]:=eta_8_T[k,i,l];
248
249         eta_16[k,i+16,l+40]:=eta_8_T[k,i,l];
250         eta_16[k,i+24,l+32]:=eta_8[k,i,l];
251       end;
252 //Construction the connection operators for n=16 from (32).
253 //1. The skew-symmetry.
254 for i:=1 to 16 do
255   for j:=1 to 128 do
256     for k:=1 to j do
257       begin
258         eta_16[i,j,k].x:=-eta_16[i,k,j].x;
259         eta_16[i,j,k].y:=-eta_16[i,k,j].y;
260         eta_16[i,j,k].x:=-eta_16[i,k,j].x;
261         eta_16[i,j,k].y:=-eta_16[i,k,j].y;
262       end;
263 //2. Duplication.
264 for i:=1 to 64 do
265   for l:=1 to 64 do
266     for k:=1 to 14 do
267       begin
268         eta_16[k,i+64,l+64]:=mul(m_,eta_16[k,l,i]);
269         eta_16[k,i+64,l+64]:=mul(m_,eta_16[k,l,i]);
270       end;
271 //3. n=15,16.
272 for i:=1 to 64 do
273   begin
274     eta_16[15,i,i+64].x:=0; eta_16[15,i,i+64].y:=1; eta_16[16,i,i+64].x:=1; eta_16[16,i,i+64].y:=0;
275     eta_16[15,i+64,i].x:=0; eta_16[15,i+64,i].y:=-1; eta_16[16,i+64,i].x:=1; eta_16[16,i+64,i].y:=0;
276     eta_16[15,i,i+64].x:=0; eta_16[15,i,i+64].y:=-1; eta_16[16,i,i+64].x:=1; eta_16[16,i,i+64].y:=0;
277     eta_16[15,i+64,i].x:=0; eta_16[15,i+64,i].y:=1; eta_16[16,i+64,i].x:=1; eta_16[16,i+64,i].y:=0;
278   end;
279

```



```

280 //4. Multiply by  $1/\sqrt{2}$ .
281 for i:=1 to 16 do
282   for j:=1 to 128 do
283     for k:=1 to 128 do
284       begin
285         eta_16[i,j,k].x:=eta_16[i,j,k].x/sqrt(2);
286         eta_16[i,j,k].y:=eta_16[i,j,k].y/sqrt(2);
287         eta_16_1[i,j,k].x:=eta_16[i,j,k].x/sqrt(2);
288         eta_16_1[i,j,k].y:=eta_16_1[i,j,k].y/sqrt(2);
289       end;
290 //Initialization the controlling spinors  $X^A$  from (34).
291   for j:=1 to 128 do
292     Init_(x[j]);
293     x[1].x:=1;
294     x[65].x:=1;
295 //Construction of the inclusion operators  $P_{jA}$  from (38).
296   for l:=1 to 4 do
297     for i:=1 to 16 do
298       for j:=1 to 128 do
299         begin
300           Init_(P[i,j]);
301           for k:=1 to 128 do
302             P[i,j]:=add(P[i,j],mul(eta_16_1[i,k,j],x[k]));
303           end;
304 //Construction of the structure constants of the generating algebra for n=16  $A_{gen}((\eta_{gen})_{ij}^k)$  from (5),(35).
305 //1. Multiply the connection operators by  $P$ .
306   begin
307     for i:=1 to 16 do
308       for j:=1 to 128 do
309         for k:=1 to 16 do
310           begin
311             eta_[i,j,k].x:=0;
312             eta_[i,j,k].y:=0;
313             for m:=1 to 128 do
314               eta_[i,j,k]:=add(eta_[i,j,k],mul(eta_16[i,j,m],P[k,m]));
315             end;
316           end;
317         for j:=1 to 16 do
318           for k:=1 to 16 do
319             begin
320               eta[i,j,k].x:=0;
321               eta[i,j,k].y:=0;
322               for m:=1 to 128 do
323                 eta[i,j,k]:=add(eta[i,j,k],mul(eta_[i,m,k],P[j,m]));
324             end;
325           end;
326 //2. Multiply by  $\sqrt{2}$ .
327   for i:=1 to 16 do
328     for j:=1 to 16 do
329       for k:=1 to 16 do
330         begin
331           eta[i,j,k].x:=eta[i,j,k].x*sqrt(2);
332           eta[i,j,k].y:=eta[i,j,k].y*sqrt(2);
333         end;
334 //Initialization the basic orthogonal transformations  $(S_I)_i^j$  ( $I=\overline{1,15}$ ) according to note 2.
335 //I=1.
336   s_orthogonal[1,3,1].x:=-1;      s_orthogonal[1,2,2].x:=1;
337   s_orthogonal[1,5,3].x:=-1;      s_orthogonal[1,4,4].x:=1;
338   s_orthogonal[1,7,5].x:=1;       s_orthogonal[1,6,6].x:=1;
339   s_orthogonal[1,9,7].x:=1;       s_orthogonal[1,8,8].x:=1;
340   s_orthogonal[1,11,9].x:=-1;     s_orthogonal[1,10,10].x:=1;
341   s_orthogonal[1,13,11].x:=1;     s_orthogonal[1,12,12].x:=1;
342   s_orthogonal[1,15,13].x:=-1;    s_orthogonal[1,14,14].x:=1;
343   s_orthogonal[1,1,15].x:=-1;     s_orthogonal[1,0,16].x:=1;
344 //I=2.
345   s_orthogonal[2,12,1].x:=-1;     s_orthogonal[2,14,2].x:=1;
346   s_orthogonal[2,1,3].x:=-1;     s_orthogonal[2,3,4].x:=1;
347   s_orthogonal[2,7,5].x:=1;       s_orthogonal[2,5,6].x:=-1;
348   s_orthogonal[2,6,7].x:=1;       s_orthogonal[2,4,8].x:=1;
349   s_orthogonal[2,11,9].x:=-1;     s_orthogonal[2,9,10].x:=-1;
350   s_orthogonal[2,10,11].x:=1;     s_orthogonal[2,8,12].x:=-1;
351   s_orthogonal[2,15,13].x:=-1;    s_orthogonal[2,13,14].x:=-1;
352   s_orthogonal[2,2,15].x:=-1;     s_orthogonal[2,0,16].x:=1;
353 //I=3.
354   s_orthogonal[3,1,1].x:=1;       s_orthogonal[3,2,2].x:=1;
355   s_orthogonal[3,7,3].x:=-1;     s_orthogonal[3,4,4].x:=1;
356   s_orthogonal[3,5,5].x:=-1;     s_orthogonal[3,6,6].x:=1;
357   s_orthogonal[3,11,7].x:=1;      s_orthogonal[3,8,8].x:=1;
358   s_orthogonal[3,9,9].x:=1;       s_orthogonal[3,10,10].x:=1;
359   s_orthogonal[3,15,11].x:=1;     s_orthogonal[3,12,12].x:=1;
360   s_orthogonal[3,13,13].x:=1;     s_orthogonal[3,14,14].x:=1;
361   s_orthogonal[3,3,15].x:=-1;     s_orthogonal[3,0,16].x:=1;
362 //I=4.
363   s_orthogonal[4,1,1].x:=1;       s_orthogonal[4,5,2].x:=-1;
364   s_orthogonal[4,3,3].x:=1;       s_orthogonal[4,7,4].x:=-1;
365   s_orthogonal[4,2,5].x:=-1;     s_orthogonal[4,6,6].x:=1;
366   s_orthogonal[4,12,7].x:=1;      s_orthogonal[4,8,8].x:=1;
367   s_orthogonal[4,9,9].x:=-1;     s_orthogonal[4,13,10].x:=1;
368   s_orthogonal[4,11,11].x:=1;     s_orthogonal[4,15,12].x:=1;
369   s_orthogonal[4,10,13].x:=1;     s_orthogonal[4,14,14].x:=1;
370   s_orthogonal[4,4,15].x:=-1;     s_orthogonal[4,0,16].x:=1;
371
372
373
374

```

```

375 //I=5.
376 s_orthogonal [5,3,1].x:=1; s_orthogonal [5,6,2].x:=1;
377 s_orthogonal [5,1,3].x:=1; s_orthogonal [5,4,4].x:=1;
378 s_orthogonal [5,13,5].x:=-1; s_orthogonal [5,8,6].x:=1;
379 s_orthogonal [5,7,7].x:=-1; s_orthogonal [5,2,8].x:=1;
380 s_orthogonal [5,9,9].x:=1; s_orthogonal [5,12,10].x:=1;
381 s_orthogonal [5,15,11].x:=-1; s_orthogonal [5,10,12].x:=1;
382 s_orthogonal [5,11,13].x:=1; s_orthogonal [5,14,14].x:=-1;
383 s_orthogonal [5,5,15].x:=-1; s_orthogonal [5,0,16].x:=1;
384 //I=6.
385 s_orthogonal [6,8,1].x:=-1; s_orthogonal [6,14,2].x:=-1;
386 s_orthogonal [6,3,3].x:=1; s_orthogonal [6,5,4].x:=-1;
387 s_orthogonal [6,4,5].x:=1; s_orthogonal [6,2,6].x:=-1;
388 s_orthogonal [6,7,7].x:=1; s_orthogonal [6,1,8].x:=1;
389 s_orthogonal [6,9,9].x:=1; s_orthogonal [6,15,10].x:=1;
390 s_orthogonal [6,11,11].x:=1; s_orthogonal [6,13,12].x:=1;
391 s_orthogonal [6,12,13].x:=1; s_orthogonal [6,10,14].x:=1;
392 s_orthogonal [6,6,15].x:=-1; s_orthogonal [6,0,16].x:=1;
393 //I=7.
394 s_orthogonal [7,1,1].x:=1; s_orthogonal [7,6,2].x:=-1;
395 s_orthogonal [7,3,3].x:=1; s_orthogonal [7,4,4].x:=1;
396 s_orthogonal [7,5,5].x:=-1; s_orthogonal [7,2,6].x:=1;
397 s_orthogonal [7,15,7].x:=1; s_orthogonal [7,8,8].x:=1;
398 s_orthogonal [7,9,9].x:=1; s_orthogonal [7,14,10].x:=-1;
399 s_orthogonal [7,11,11].x:=1; s_orthogonal [7,12,12].x:=-1;
400 s_orthogonal [7,13,13].x:=1; s_orthogonal [7,10,14].x:=1;
401 s_orthogonal [7,7,15].x:=-1; s_orthogonal [7,0,16].x:=1;
402 //I=8.
403 s_orthogonal [8,1,1].x:=1; s_orthogonal [8,9,2].x:=-1;
404 s_orthogonal [8,3,3].x:=1; s_orthogonal [8,11,4].x:=-1;
405 s_orthogonal [8,5,5].x:=1; s_orthogonal [8,13,6].x:=-1;
406 s_orthogonal [8,7,7].x:=1; s_orthogonal [8,15,8].x:=1;
407 s_orthogonal [8,2,9].x:=1; s_orthogonal [8,10,10].x:=1;
408 s_orthogonal [8,4,11].x:=1; s_orthogonal [8,12,12].x:=-1;
409 s_orthogonal [8,6,13].x:=1; s_orthogonal [8,14,14].x:=-1;
410 s_orthogonal [8,8,15].x:=-1; s_orthogonal [8,0,16].x:=1;
411 //I=9.
412 s_orthogonal [9,1,1].x:=1; s_orthogonal [9,8,2].x:=1;
413 s_orthogonal [9,3,3].x:=1; s_orthogonal [9,10,4].x:=1;
414 s_orthogonal [9,5,5].x:=1; s_orthogonal [9,12,6].x:=1;
415 s_orthogonal [9,11,7].x:=-1; s_orthogonal [9,2,8].x:=1;
416 s_orthogonal [9,13,9].x:=-1; s_orthogonal [9,4,10].x:=1;
417 s_orthogonal [9,15,11].x:=-1; s_orthogonal [9,6,12].x:=1;
418 s_orthogonal [9,7,13].x:=1; s_orthogonal [9,14,14].x:=-1;
419 s_orthogonal [9,9,15].x:=-1; s_orthogonal [9,0,16].x:=1;
420 //I=10.
421 s_orthogonal [10,1,1].x:=1; s_orthogonal [10,11,2].x:=1;
422 s_orthogonal [10,3,3].x:=1; s_orthogonal [10,9,4].x:=-1;
423 s_orthogonal [10,5,5].x:=1; s_orthogonal [10,15,6].x:=-1;
424 s_orthogonal [10,7,7].x:=1; s_orthogonal [10,13,8].x:=-1;
425 s_orthogonal [10,4,9].x:=1; s_orthogonal [10,14,10].x:=1;
426 s_orthogonal [10,2,11].x:=1; s_orthogonal [10,8,12].x:=1;
427 s_orthogonal [10,12,13].x:=-1; s_orthogonal [10,6,14].x:=1;
428 s_orthogonal [10,10,15].x:=-1; s_orthogonal [10,0,16].x:=1;
429 //I=11.
430 s_orthogonal [11,1,1].x:=1; s_orthogonal [11,10,2].x:=-1;
431 s_orthogonal [11,3,3].x:=1; s_orthogonal [11,8,4].x:=1;
432 s_orthogonal [11,5,5].x:=1; s_orthogonal [11,14,6].x:=1;
433 s_orthogonal [11,15,7].x:=-1; s_orthogonal [11,4,8].x:=1;
434 s_orthogonal [11,9,9].x:=1; s_orthogonal [11,2,10].x:=1;
435 s_orthogonal [11,7,11].x:=1; s_orthogonal [11,12,12].x:=1;
436 s_orthogonal [11,13,13].x:=1; s_orthogonal [11,6,14].x:=1;
437 s_orthogonal [11,11,15].x:=-1; s_orthogonal [11,0,16].x:=1;
438 //I=12.
439 s_orthogonal [12,1,1].x:=1; s_orthogonal [12,13,2].x:=1;
440 s_orthogonal [12,3,3].x:=1; s_orthogonal [12,15,4].x:=1;
441 s_orthogonal [12,5,5].x:=1; s_orthogonal [12,9,6].x:=-1;
442 s_orthogonal [12,7,7].x:=1; s_orthogonal [12,11,8].x:=1;
443 s_orthogonal [12,6,9].x:=1; s_orthogonal [12,10,10].x:=1;
444 s_orthogonal [12,8,11].x:=-1; s_orthogonal [12,4,12].x:=1;
445 s_orthogonal [12,2,13].x:=1; s_orthogonal [12,14,14].x:=1;
446 s_orthogonal [12,12,15].x:=-1; s_orthogonal [12,0,16].x:=1;
447 //I=13.
448 s_orthogonal [13,1,1].x:=1; s_orthogonal [13,12,2].x:=-1;
449 s_orthogonal [13,3,3].x:=1; s_orthogonal [13,14,4].x:=-1;
450 s_orthogonal [13,5,5].x:=1; s_orthogonal [13,8,6].x:=1;
451 s_orthogonal [13,11,7].x:=1; s_orthogonal [13,6,8].x:=1;
452 s_orthogonal [13,7,9].x:=1; s_orthogonal [13,10,10].x:=1;
453 s_orthogonal [13,15,11].x:=-1; s_orthogonal [13,2,12].x:=1;
454 s_orthogonal [13,9,13].x:=-1; s_orthogonal [13,4,14].x:=1;
455 s_orthogonal [13,13,15].x:=-1; s_orthogonal [13,0,16].x:=1;
456 //I=14.
457 s_orthogonal [14,1,1].x:=1; s_orthogonal [14,15,2].x:=-1;
458 s_orthogonal [14,3,3].x:=1; s_orthogonal [14,13,4].x:=1;
459 s_orthogonal [14,5,5].x:=1; s_orthogonal [14,11,6].x:=-1;
460 s_orthogonal [14,7,7].x:=1; s_orthogonal [14,9,8].x:=-1;
461 s_orthogonal [14,8,9].x:=1; s_orthogonal [14,6,10].x:=1;
462 s_orthogonal [14,10,11].x:=-1; s_orthogonal [14,4,12].x:=1;
463 s_orthogonal [14,12,13].x:=1; s_orthogonal [14,2,14].x:=1;
464 s_orthogonal [14,14,15].x:=-1; s_orthogonal [14,0,16].x:=1;
465
466
467
468
469
470

```

```

471 //I=15.
472 s_orthogonal[15,1,1].x:=1; s_orthogonal[15,14,2].x:=1;
473 s_orthogonal[15,3,3].x:=1; s_orthogonal[15,12,4].x:=-1;
474 s_orthogonal[15,5,5].x:=1; s_orthogonal[15,10,6].x:=1;
475 s_orthogonal[15,7,7].x:=1; s_orthogonal[15,8,8].x:=-1;
476 s_orthogonal[15,9,9].x:=-1; s_orthogonal[15,6,10].x:=1;
477 s_orthogonal[15,11,11].x:=-1; s_orthogonal[15,4,12].x:=1;
478 s_orthogonal[15,13,13].x:=1; s_orthogonal[15,2,14].x:=1;
479 s_orthogonal[15,15,15].x:=-1; s_orthogonal[15,0,16].x:=1;
480 //Constructing the structural constant of the canonical sedenion algebra for n=16.
481 //1. Constructing the structural constants of the basic algebras
482 //A_I : (η_I)_{ij}^k := (S_I)_i^l (S_I)_j^m (η_{gen})_{lm}^r (S_I)^k_r (I = 1,15) from (29).
483 for m:=1 to 15 do
484 begin
485 for i:=1 to 16 do
486 for j:=1 to 16 do
487 for k:=0 to 15 do
488 begin
489 eta_orthogonal[m,i,j,k].x:=0;
490 eta_orthogonal[m,i,j,k].y:=0;
491 for l:=1 to 16 do
492 eta_orthogonal[m,i,j,k]:=
493 add(eta_orthogonal[m,i,j,k],mul(s_orthogonal[m,k,l],eta[i,j,l]));
494 end;
495 for i:=1 to 16 do
496 for j:=0 to 15 do
497 for k:=0 to 15 do
498 begin
499 eta_orthogonal_[m,i,j,k].x:=0;
500 eta_orthogonal_[m,i,j,k].y:=0;
501 for l:=1 to 16 do
502 eta_orthogonal_[m,i,j,k]:=
503 add(eta_orthogonal_[m,i,j,k],mul(s_orthogonal[m,j,l],eta_orthogonal[m,i,l,k]));
504 end;
505 for i:=0 to 15 do
506 for j:=0 to 15 do
507 for k:=0 to 15 do
508 begin
509 eta_orthogonal[m,i,j,k].x:=0;
510 eta_orthogonal[m,i,j,k].y:=0;
511 for l:=1 to 16 do
512 eta_orthogonal[m,i,j,k]:=
513 add(eta_orthogonal[m,i,j,k],mul(s_orthogonal[m,i,l],eta_orthogonal_[m,l,j,k]));
514 end;
515 end;
516 //2. Initialization the complex facror -12(m_12), 1/3(m_3).
517 m_12.x:=-12;
518 m_12.y:=0;
519 m_3.x:=1/3;
520 m_3.y:=0;
521 //3. Constructing according to
522 //η_{ij}^k := (1 - 15 ∑_{I=1} 1/3 I(η_0)_{ij}^k + 1/3 ∑_{I=1}^{15} ((h_I)_{ij}^k + (η_0)_{ij}^k) = 1/3(-12(η_0)_{ij}^k + ∑_{I=1}^{15} (η_I)_{ij}^k),
523 // (η_0)_{ij}^k := ((1/√2)η_i)δ_j^k + (1/√2)η_jδ_i^k - g_{ij}(1/√2)η^k from (29),(41).
524 for i:=0 to 15 do
525 for j:=0 to 15 do
526 for k:=0 to 15 do
527 begin
528 eta_constant[i,j,k].x:=0;
529 eta_constant[i,j,k].y:=0;
530 for m:=1 to 15 do
531 eta_constant[i,j,k]:=add(eta_constant[i,j,k],eta_orthogonal[m,i,j,k]);
532 eta_constant[i,j,k]:=mul(m_3,add(eta_constant[i,j,k],
533 mul(m_12,add(add(mul(unit_[j],g[i,k]),mul(unit_[i],g[j,k])),mul(m_,mul(g[i,j],unit_[k])))))));
534 end;
535 //Output of the sedenion multiplication table.
536 for i:=1 to 16 do
537 begin
538 StringGrid1.Cells[0,i]:=IntToStr(i-1);
539 StringGrid1.Cells[i,0]:=IntToStr(i-1);
540 end;
541 for i:=0 to 15 do
542 for j:=0 to 15 do
543 begin
544 StringGrid1.Cells[j+1,i+1]:='';
545 for k:=0 to 15 do
546 if Round(eta_constant[i,j,k].x)<>0 then
547 StringGrid1.Cells[j+1,i+1]:=IntToStr(Round(eta_constant[i,j,k].x))+*e'+IntToStr(k);
548 end;
549 //Output of the sedenion multiplication table into the file.
550 IniFile:=TIniFile.Create(GetCurrentDir+'\IniFile.ini');
551 array_str:='';
552 for i:=0 to 15 do
553 begin
554 array_str:=array_str+Format('%12s',[chr(36)+e_{'+IntToStr(i)+'}'+chr(36)]);
555 if i<15 then
556 array_str:=array_str+'_';
557 end;
558
559
560
561
562

```

```

563   IniFile.WriteString('Вариант_', '0000000*', array_str+'\\_\\hline');
564   for i:=0 to 15 do
565   begin
566     array_str:='';
567     for j:=0 to 15 do
568     begin
569       for k:=0 to 15 do
570         if Round(eta_constant[i,j,k].x)<>0 then
571           array_str:=array_str+Format('%12s',[IntToStr(Round(eta_constant[i,j,k].x))+
572
573             'x'+chr(36)+'e_'+IntToStr(k)+''+chr(36)]]);
574         if j<15 then
575           array_str:=array_str+'_&_';
576       end;
577     if i<10 then
578       IniFile.WriteString('Вариант_', '0'+chr(36)+'e_'+IntToStr(i)+''+chr(36), array_str+'\\_\\hline')
579     else
580       IniFile.WriteString('Вариант_', chr(36)+'e_'+IntToStr(i)+''+chr(36), array_str+'\\_\\hline');
581   end;
582 end;
583 end.
584 //

```

References

- [1] A.A. ALBERT Quadratic Forms Permitting Composition. Ann. of Math. 1942, 43, 161-177.
- [2] АНДРЕЕВ К. В. О спинорном формализме при четной размерности базового пространства. ВИНТИ - 298-B-11, июнь 2011. 138 с. [K.V. Andreev. On the spinor formalism for the base space of even dimension. VINITI-298-V-11, Jun 2011. 138pp. Paper deponed on Jun 16, 2011 at VINITI (Moscow), ref. №298-V 11]
- [3] АНДРЕЕВ К. В. О спинорном формализме при четном n . arXiv:1110.4737v2 [math-ph] [K.V. Andreev. On the spinor formalism for even n . arXiv:1110.4737v2 [math-ph]]
- [4] БАЭЗ ДЖОН С. Октонионы.// Гиперкомплексные числа в геометрии и физике. №1(5), Vol. 3 (2006), с.120-177 [John C. Baez The Octonions.//ArXiv:math.RA/0105155 v4 23 Apr 2002]
- [5] R. GUILLERMO MORENO The zero divisors of the Cayley-Dickson algebras over the real numbers. arXiv:q-alg/9710013v1 [math.QA]
- [6] М.М.ПОСТНИКОВ Лекции по геометрии. V семестр. Группы и алгебры Ли. Москва, Наука, 1982 г. [M. Postnikov, Lie Groups and Lie Algebras. Lectures in Geometry. Semester V, Mir, Moscow, 1986. MR 0905471 (88f:22002)].